

ON HELLY NUMBER FOR CRYSTALS AND CUT-AND-PROJECT SETS

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ABSTRACT. We prove existence of Helly numbers for crystals and for cut-and-project sets with convex window. Also we show that for a two-dimensional crystal consisting of k copies of a single lattice the Helly number does not exceed $k + 6$.

1. INTRODUCTION

The Helly theorem [11] states that if any $d + 1$ or fewer sets from a finite family \mathcal{F} of convex sets in \mathbb{R}^d have non-empty intersection, then all sets in \mathcal{F} have non-empty intersection. In other words, it claims that the Helly number of \mathbb{R}^d is $d + 1$.

Similarly, the following theorem claims that the Helly number of the d -dimensional lattice \mathbb{Z}^d is 2^d .

Theorem 1.1 (J.-P. Doignon, [9]). *Let \mathcal{F} be a finite family of convex sets in \mathbb{R}^d . If any 2^d sets from \mathcal{F} intersect at a point of \mathbb{Z}^d , then all sets from \mathcal{F} intersect at a point of \mathbb{Z}^d .*

This theorem is generalized in the following fractional version.

Theorem 1.2 (I. Bárány, J. Matoušek, [5]). *For every $d \geq 1$ and every $\alpha \in (0, 1]$ there exists a $\beta = \beta(d, \alpha) > 0$ with the following property. Let \mathcal{F} be a family of n convex sets in \mathbb{R}^d such that at least $\alpha \binom{n}{d+1}$ subfamilies of $d + 1$ sets intersect at a point from \mathbb{Z}^{d+1} . Then there exists a point of \mathbb{Z}^d contained in at least βn sets from \mathcal{F} .*

The goal of this paper is to prove Helly-type theorems (usual and fractional) for crystals and cut-and-project sets (that in certain cases can be considered as mathematical models of quasicrystals), the discrete point set defined in the section 2.

This paper is organized as follows. In the section 2 we introduce main definitions. In the section 3 we prove existence of a Helly number for every d -dimensional crystal and for every cut-and-project set with convex window. Finally, in the section 4 we give exact maximum value of the Helly number for a two-dimensional k -crystal, i.e. k translational copies of a fixed two-dimensional lattice.

2. HELLY NUMBERS FOR ARBITRARY SETS

First of all we introduce the S -Helly number of an arbitrary set $S \subset \mathbb{R}^d$.

Definition 2.1. Let S be a non-empty subset of \mathbb{R}^d . We define the S -Helly number, or $h(S)$, to be the minimal number n such that the following statement holds. If for a finite family \mathcal{F} of convex sets in \mathbb{R}^d with at least n elements any n sets from \mathcal{F} intersect at a point of S , then all sets from \mathcal{F} intersect at a point of S .

In particular the classical Helly theorem means that $h(\mathbb{R}^d) = d + 1$ and Doignon's theorem means that $h(\mathbb{Z}^d) = 2^d$. More result of Helly numbers for various sets can be found in [1, 8].

It is not very hard to see that for arbitrary set S its Helly number not necessary exists (or we can say that $h(S) = \infty$) even if S is a discrete set. If for any $n \geq 3$ S has n points

forming a convex n -gon without points of S inside, then the following lemma shows that $h(S) \geq n$ for any prescribed n , so $h(S)$ can't be finite.

Lemma 2.2 (G. Averkov, [2, Thm. 2.1]). *Assume $S \subset \mathbb{R}^d$ is discrete, then $h(S)$ is equal to the following two numbers:*

- (1) *The supremum of the number of facets of a convex polytope P such that each facet of P contains exactly one point from S in its relative interior, and no other points of S are contained in P .*
- (2) *The supremum of the number of vertices of a convex polytope Q with vertices in S that does not contain any other points from S .*

One can construct such S to be even a Delone set, i.e. a discrete set in \mathbb{R}^d such that every sufficiently large ball in \mathbb{R}^d contains at least one point from this set and every sufficiently small ball in \mathbb{R}^d contains at most one point from this set. To do this in \mathbb{R}^2 with usual (x, y) -coordinate system (this example can be easily generalized to \mathbb{R}^d for any $d \geq 2$) we take the lattice \mathbb{Z}^2 and for every $n \geq 3$ add a thin n -gon “almost” on the line $x = \frac{1}{2} + n$.

The main goal of this paper is to study Helly numbers of periodic and some quasiperiodic Delone sets.

3. CRYSTALS AND CUT-AND-PROJECT SETS

Definition 3.1. Let Λ a d -dimensional lattice in \mathbb{R}^d . Any union of finite translates of Λ is called a *d -dimensional crystal*.

To be more precise, if $\mathbf{t}_1, \dots, \mathbf{t}_k$ are vectors in \mathbb{R}^d such that for every $i \neq j$ difference $\mathbf{t}_i - \mathbf{t}_j \notin \Lambda$, then the set

$$\bigcup_{i=1}^k \{\Lambda + \mathbf{t}_i\}$$

is called a *d -dimensional k -crystal*.

Crystals are the only periodic Delone sets in \mathbb{R}^d , and by periodic set in \mathbb{R}^d we mean a set which symmetry group posses d linearly independent translations.

Existence of Helly number follows from the following lemma [1, Prop. 2.8].

Lemma 3.2. *If $S_1, S_2 \subseteq \mathbb{R}^2$, then $h(S_1 \cup S_2) \leq h(S_1) + h(S_2)$.*

Corollary 3.3. *If S is a d -dimensional k -crystal, then $h(S) \leq k2^d$.*

Proof. It is clear that Helly number does not change if we apply an affine transformation to a set, and thus for any d -dimensional lattice Λ $h(\Lambda) = 2^d$.

Since S can be written as $\bigcup_{i=1}^k \{\Lambda + \mathbf{t}_i\}$, the lemma 3.2 implies that

$$h(S) \leq \sum_{i=1}^k h(\Lambda + \mathbf{t}_i) = \sum_{i=1}^k 2^d = k2^d.$$

□

In the section 4 that this trivial bound is not sharp at least in the case of two-dimensional crystals.

The other class of Delone we will be interested in is the class of cut-and-project sets.

Definition 3.4. Let \mathbb{R}^{d+k} be represented as $\mathbb{R}^{d+k} = \mathbb{R}^d \times \mathbb{R}^k$ and let Λ be a lattice in \mathbb{R}^{d+k} . Let $\pi_1 : \mathbb{R}^{d+k} \rightarrow \mathbb{R}^d$ and $\pi_2 : \mathbb{R}^{d+k} \rightarrow \mathbb{R}^k$ be two projections on complementary

subspaces such that $\pi_1|_\Lambda$ is injective, and $\pi_2(\Lambda)$ is dense in \mathbb{R}^k . Let $W \subset \mathbb{R}^k$ be a compact set — the *window* — such that the closure of the interior of W equals W .

Then

$$V := V(\mathbb{R}^d, \mathbb{R}^k, \Lambda, W) = \{\pi_1(\mathbf{x}) \mid \mathbf{x} \in \Lambda, \pi_2(\mathbf{x}) \in W\}$$

is called a *cut-and-project set*, or a *model set*.

This is summarised in the following diagram, which is called *cut-and-project scheme*.

$$\begin{array}{ccccc} \mathbb{R}^d & \xleftarrow{\pi_1} & \mathbb{R}^d \times \mathbb{R}^k & \xrightarrow{\pi_2} & \mathbb{R}^k \\ \cup & & \cup & & \cup \\ V & & \Lambda & & W \end{array}$$

If $\mu(\partial(W)) = 0$, then V is called *regular* cut-and-project set.

If $\partial(W) \cap \pi_2(\Gamma) = \emptyset$, then V is called *generic* cut-and-project set.

This definition can be generalized for locally compact abelian groups G and H instead of \mathbb{R}^d and \mathbb{R}^k . See [12] for survey on model sets and more references.

Though the definition of cut-and-project sets is pretty technical, as first approximation one can construct certain cut-and-project sets in the following way. Let $n = d + k$. In \mathbb{R}^n we can take a d -dimensional affine subspace γ with irrational slope, that is no vector with rational coordinates is parallel to γ . For any $\epsilon > 0$ the ϵ -neighborhood of γ will contain points from \mathbb{Z}^d , moreover orthogonal projections of these points on γ will form a Delone set which will be exactly $V(\gamma, \gamma^\perp, \mathbb{Z}^n, W)$ where γ^\perp is any orthogonal complement of γ , and W is the ϵ -ball centered at $\gamma \cap \gamma^\perp$.

The resulted cut-and-project set will not be periodic because of our choice of γ with irrational slope, but it will be *quasiperiodic*, meaning that every local picture in V will repeat infinitely many times. This allows to use cut-and-project sets as model for mathematical quasicrystals. See [4] for more details.

Some examples of quasiperiodic cut-and-project sets that can be obtained in the way described in the definition 3.4 include Penrose tilings [7], Ammann-Beenker tilings [6], and many other, see [10] for example. Certain local patches of Penrose and Ammann-Beenker tilings are shown on the figure 1. To be precise, the Penrose tiling is not a cut-and-project defined in the definition 3.4 because the corresponding projection π_2 will not give the dense image in the corresponding window (the image will be dense in four two-dimensional planes inside a three-dimensional polytope, see [7]). However, such a “degeneracy” if one occur, will not affect our proof, so we will not emphasize separately whether image of π_2 is dense or not.

Note, that here we substituted “Delone set” with “tiling”. In the case the set of vertices of a tiling form a corresponding Delone set. Tilings here provide more visual approach to present Delone sets as in addition to just points, they also have some “structure” between them.

Our next goal is to prove existence of Helly number for any cut-and-project set with convex window.

Theorem 3.5. *Let $V = V(\mathbb{R}^d, \mathbb{R}^k, \Lambda, W)$ be a d -dimensional cut-and-project set with convex k -dimensional window W . Then $h(V) \leq 2^{d+k}$.*

Proof. Let \mathcal{F} be a finite family of convex d -dimensional sets such that every 2^{d+k} sets from \mathcal{F} intersect at a point of V . We will show that all sets from \mathcal{F} intersect at a point of V which will be enough to prove the theorem.

Let π_1 and π_2 be the projections used in the constructing of V . As $\pi_1^{-1}(X_1)$ and $\pi_2^{-1}(X_2)$ we will denote complete preimages of X_1 and X_2 with respect to corresponding projection.

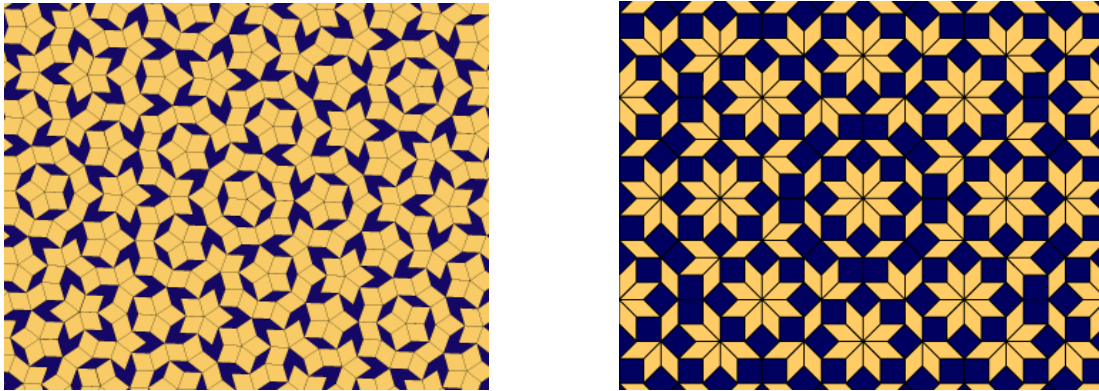


FIGURE 1. Penrose and Ammann-Beenker tilings. The original pictures can be found at [10].

Let $\mathcal{F}' = \{\pi_1^{-1}(F) \cap \pi_2^{-1}(W) | F \in \mathcal{F}\}$ be the family of intersections of preimages of elements of \mathcal{F} with respect to projection π_1 with preimage of W with respect to projection π_2 . Since W is convex and the complete preimage of a convex set with respect to any projection is convex, \mathcal{F}' is a finite family of convex sets in \mathbb{R}^{d+k} .

Let $F'_1, \dots, F'_{2^{d+k}}$ be any sets from \mathcal{F}' . Their projections $\pi_1(F'_1), \dots, \pi_1(F'_{2^{d+k}})$ are sets from \mathcal{F} so they intersect at a point $\mathbf{x} \in V$. Due to construction of V , there is a (unique) point $\mathbf{x}' \in \Lambda$ such that $\pi_1(\mathbf{x}') = \mathbf{x}$ and $\pi_2(\mathbf{x}') \in W$. Therefore \mathbf{x}' belongs to every F'_i for $i = 1, \dots, 2^{d+k}$, and every 2^{d+k} sets from \mathcal{F}' intersect at a point of Λ . Since $h(\Lambda) = 2^{d+k}$, all sets from \mathcal{F}' intersect at a point $\mathbf{x}'_0 \in \Lambda$.

The projection $\mathbf{x}_0 = \pi_1(\mathbf{x}'_0)$ is a point of V because $\pi_2(\mathbf{x}'_0) \in W$. This projection \mathbf{x}_0 belongs to every set from \mathcal{F} , therefore all set from \mathcal{F} intersect at a point of V . \square

Corollary 3.6. *If P is the set of vertices of a Penrose tiling, then $h(P) \leq 32$.*

Proof. Sections 7 and 8 of [7] establish that vertices of a Penrose tiling form a two-dimensional cut-and-project set with three-dimensional window. Applying the theorem 3.5, we get $h(P) \leq 2^{2+3} = 32$. \square

Corollary 3.7. *Convex hull of any 33 vertices of a Penrose tiling has a vertex of the same tiling inside.*

Proof. The statement immediately follows from the previous corollary and the second part of the lemma 2.2. \square

Referring to the theorem 3.5 and lemma 2.2 we can prove analogous result for any cut-and-project set with convex window.

Corollary 3.8. *If V is a d -dimensional cut-and-project set with a convex k -dimensional window, then any convex polytope with at least $2^{d+k} + 1$ vertices of V has a vertex of V inside.*

The theorem [3, Thm. 1.3] states that once Helly number of a closed subset $S \subseteq \mathbb{R}^d$ is finite, then there is a fractional version of the Helly theorem for S , and the fractional Helly constant is at most $d + 1$. The corollary 3.3 and theorem 3.5 implies that this can be applied to crystals and cut-and-project sets as both classes consist of closed sets.

Corollary 3.9. *Let $S \subset \mathbb{R}^d$ be a crystal or a cut-and-project set.*

For every $\alpha \in (0, 1]$ there exists a $\beta = \beta(S, \alpha) > 0$ with the following property. Let \mathcal{F} be a family of n convex sets in \mathbb{R}^d such that at least $\alpha \binom{n}{d+1}$ subfamilies of $d + 1$ sets intersect at a point from S . Then there exists a point of S contained in at least βn sets from \mathcal{F} .

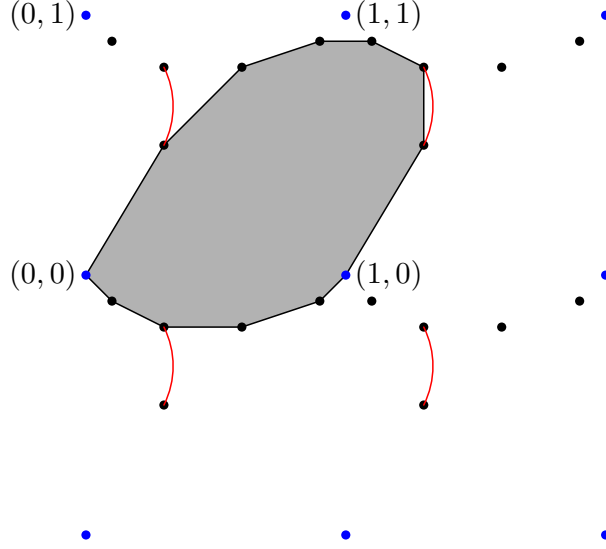


FIGURE 2. Two-dimensional k -crystal with Helly number $k + 6$.

4. HELLY NUMBER FOR TWO-DIMENSIONAL CRYSTALS

In the last section we give sharp bound for Helly number for two-dimensional crystals. The corollary 3.3 says that the Helly number of a two-dimensional k -crystal is not greater than $4k$. Below we will show that it is not greater than $k+6$ for $k \geq 6$, and even smaller for other k . Without loss of generality for all crystals below we will use \mathbb{Z}^2 as the generating lattice.

We start from an example that Helly constant $k + 6$ can be attained. Throughout the section we refer to the second part of the lemma 2.2, so for a given crystal S we will be interested only in empty convex polygons with vertices from S with maximal number of vertices.

Example 4.1. First we take the 6-crystal consisting of translations of \mathbb{Z}^2 by vectors $(0,0), (\frac{3}{10}, \frac{5}{10}), (\frac{6}{10}, \frac{8}{10}), (\frac{9}{10}, \frac{9}{10}), (\frac{11}{10}, \frac{9}{10}), (\frac{13}{10}, \frac{8}{10})$.

One can see on the figure 2, that the corresponding 6-crystal has an empty 12-gon with vertices $(0,0), (\frac{3}{10}, \frac{5}{10}), (\frac{6}{10}, \frac{8}{10}), (\frac{9}{10}, \frac{9}{10}), (\frac{11}{10}, \frac{9}{10}), (\frac{13}{10}, \frac{8}{10}), (\frac{13}{10}, \frac{5}{10}), (1,0), (\frac{9}{10}, -\frac{1}{10}), (\frac{6}{10}, -\frac{2}{10}), (\frac{4}{10}, -\frac{2}{10}), (\frac{1}{10}, -\frac{1}{10})$. The integer points are colored with blue for more clarity of the picture.

If we want to construct a crystal with more than 6 copies of \mathbb{Z}^2 , then we can add any number of additional copies translated by vectors represented by points on the red circular arc connecting points $(\frac{13}{10}, \frac{8}{10})$ and $(\frac{13}{10}, \frac{5}{10})$ on the figure 2. Note, that if we add $k - 6$ additional points (so the resulting crystal will be a k -crystal), then we will be able to find an empty convex $(k + 6)$ -gon (12 initial points and $k - 6$ points on the red arc), so the resulting crystal will have the Helly number at least $k + 6$.

In the remaining part of the paper we will show, that the previous example is optimal, if there are at least 6 copies of \mathbb{Z}^2 , so we will prove the following theorem. To make the paper easier to read we will present proofs of some geometrical lemmas in appendix.

Theorem 4.2. *Let S be a 2-dimensional k -crystal. Then $h(S) \leq k + 6$.*

Proof. Let A_1, A_2, \dots, A_n be points in S with maximal possible n such that $P = \text{conv}(A_1, A_2, \dots, A_n)$ is a convex polygon with n vertices that does not have additional

points of S inside or on the boundary. According to lemma 2.2, it is enough to show that $n \leq k + 6$.

Let N be the maximal number of vertices of P that belong to one copy of \mathbb{Z}^2 . We will study all possible cases of N and find a bound for n for each case.

Case 1: $n \geq 5$. This case is impossible because according to the lemma 2.2 the convex hull of any 5-gon with vertices at integer points will have an integer point inside or on the boundary (because $h(\mathbb{Z}^2) = 4$).

Case 2: $n = 4$. The proof of the following lemma can be found in the appendix.

Lemma 4.3. *Every convex quadrilateral Q is either parallelogram, or contains a parallelogram Q' such that three vertices of Q' are vertices of Q .*

Let A, B, C, D be four vertices of P from one copy of \mathbb{Z}^2 . If $\text{conv}(A, B, C, D)$ is not a parallelogram, then the fourth vertex of the parallelogram from lemma 4.3 is also from the same copy of \mathbb{Z}^2 and P is not empty.

If $\text{conv}(A, B, C, D)$ is a parallelogram, then it is a fundamental parallelogram of \mathbb{Z}^2 because it is empty, and thus it will have a point from any translation of \mathbb{Z}^2 inside or on the boundary. Therefore, if $k \geq 2$, then this case is impossible. We will collect maximal values of n for various k in the following table:

Case 2 : $N = 4$	k	1	2	3	4	5	≥ 6
	max. n	4	—	—	—	—	—

Case 3: $n = 3$. Let A, B, C be three vertices of P from one copy of \mathbb{Z}^2 . The area of triangle ABC must be $\frac{1}{2}$, or it will be non-empty otherwise. Applying affine transformation we can make $A = (0, 0)$, $B = (1, 0)$ and $C = (1, 1)$ as any lattice triangle with area $\frac{1}{2}$ can be transformed into this one.

Assume P has a vertex $M = (x, y)$ with $y > 1$. Then triangle ABM is not empty because it is either contains the integer point $(\lfloor \frac{x}{y} \rfloor + 1, 1)$ or the point $(x - \lfloor \frac{x}{y} \rfloor, y - 1) \in M + \mathbb{Z}^2$. Similarly P can't have a vertex $M = (x, y)$ with $x < 0$ (then BCM will be non-empty) or with $x - y > 1$ (then CAM will be non-empty). Therefore, all vertices of P are inside triangle with vertices $(0, -1)$, $(2, 1)$, $(0, 1)$.

Assume there is one more copy of \mathbb{Z}^2 which contains three vertices of P . Then these vertices should be $D = (x, y)$, $E = (x + 1, y)$, and $F = (x, y - 1)$, see the figure 3.

Now we can see that translations of the hexagon $AFBECD$ cover the plane, and there could not be other copies of \mathbb{Z}^2 in the crystal S . Also P can't have more than 6 vertices, as all points of S inside the triangle with vertices $(0, -1)$, $(2, 1)$, $(0, 1)$ are already vertices of P .

For the rest of this case all other copies of \mathbb{Z}^2 have not more than 2 points among vertices of P . Note, that if a copy of \mathbb{Z}^2 has two points among vertices of P , then the vector connecting these vertices should be $(0, 1)$, $(1, 0)$, or $(1, 1)$, otherwise both points can't be inside the triangle with vertices $(0, -1)$, $(2, 1)$, $(0, 1)$. This, together with the following lemma (the proof can be found in the appendix), will let us bound the number of copies of \mathbb{Z}^2 with two points among vertices of P .

Lemma 4.4. *Let XX' , YY' and ZZ' be three equal and parallel segments. Then $\text{conv}(X, Y, Z, X', Y', Z')$ contains at least one of these points inside or on the boundary.*

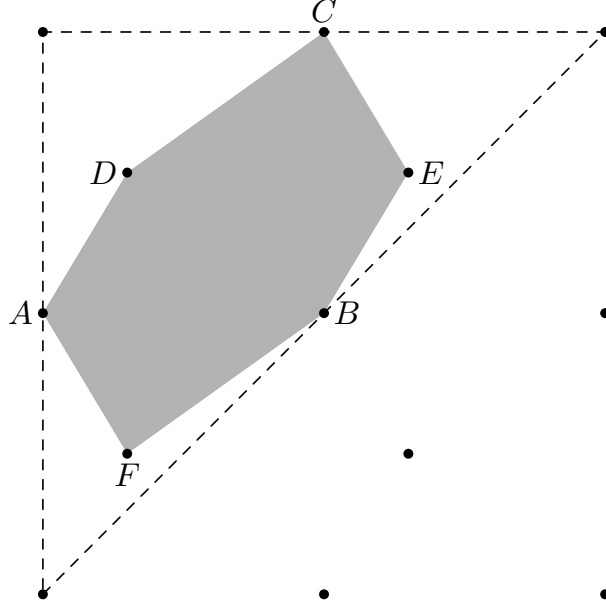


FIGURE 3. P has three vertices from two copies of \mathbb{Z}^2 .

If there are at least four additional copies of \mathbb{Z}^2 with two points among vertices of P , then at least two corresponding pairs of segments are equal and parallel, and applying the lemma 4.4 to these four points and parallel side of triangle ABC we get a contradiction. Thus, if $k \geq 4$, then all copies of \mathbb{Z}^2 , except possibly four, can have at most one point among vertices of P . If k is 3, then one copy can have three points, and two other can have two points each.

The third case can be summarized in the following table:

Case 3 : $N = 3$	k	1	2	3	4	5	≥ 6
	max. n	3	6	7	9	10	$k + 5$

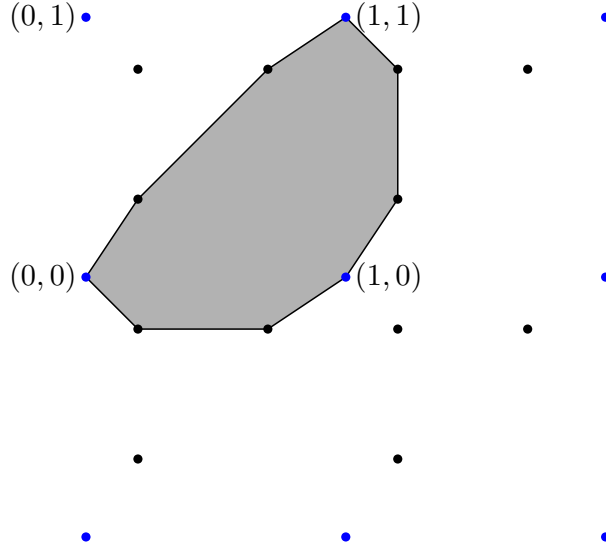
Case 4: $n = 2$. Assume that there are more than 7 copies of \mathbb{Z}^2 that have two points among vertices of P each. Note, that a vector connecting two vertices of P from the same copy of \mathbb{Z}^2 must be a basis vector of \mathbb{Z}^2 . According to lemma 4.4, there could be at most two vectors of any direction, so there will be vectors of at most four directions.

Assume there among these four vectors there is a pair which does not form a basis of \mathbb{Z}^2 . Without loss of generality we may assume that points $(0, 0)$ and $(1, 0)$ form one vector, and the second vector is formed by points (x, y) and (x', y') with $|y - y'| \geq 2$. Also, without loss of generality we may assume that $y \geq 1$. Similarly to argument in the third case, the polygon P will not be empty as it will contain the point $(\lfloor \frac{x}{y} \rfloor + 1, 1)$ or the point $(x - \lfloor \frac{x}{y} \rfloor, y - 1)$.

Now we will show that it is impossible to have four vectors in \mathbb{R}^2 such that each pair form a basis of \mathbb{Z}^2 . Without loss of generality we can assume that one pair is the standard basis $(1, 0)$ and $(0, 1)$. Then each coordinate of remaining vectors should be 1 or -1 , but any two vectors of the form $(\pm 1, \pm 1)$ do not form a basis of \mathbb{Z}^2 (the corresponding determinant will be 0 or ± 2).

Thus, the table for the case 3 is the following (case $k = 1$ is impossible here, as P can't be a 2-gon):

Case 4 : $N = 2$	k	1	2	3	4	5	≥ 6
	max. n	—	4	6	8	10	$k + 6$

FIGURE 4. 4-crystal with $h(S) = 9$.

Case 5: $n = 1$. In that case we can write the table immediately:

Case 5 : $N = 1$	k	1	2	3	4	5	≥ 6
max. n	—	—	3	4	5	k	

Summarizing all tables we can see that $h(S) \leq k + 6$ for every k . \square

In addition to the previous theorem we can find sharp bound for $h(S)$ for every k .

If $k \geq 6$ then $h(S) \leq k + 6$ and the example with $h(S) = k + 6$ for any $k \geq 6$ is described in the example 4.1. If $k = 5$, then $h(S) \leq 10$ and for equality we can take the example 4.1 and remove one cope of \mathbb{Z}^2 . If $k = 4$ then $h(S) \leq 9$ and the example with equality consist of \mathbb{Z}^2 shifted by $(0, 0)$, $(\frac{2}{10}, \frac{3}{10})$, $(\frac{7}{10}, \frac{8}{10})$, $(\frac{12}{10}, \frac{8}{10})$, see the figure 4. If $k = 3$, then $h(S) \leq 7$ and for example with equality we can take the crystal for $k = 4$ and remove the copy of \mathbb{Z}^2 translated by $(\frac{12}{10}, \frac{8}{10})$. If $k = 2$, then $h(S) \leq 6$ with example shown on the figure 3. Finally, if $k = 1$ then S is a lattice and $h(S) = 4$.

Altogether this can be summarized in the following table:

k	1	2	3	4	5	≥ 6
max. n	4	6	7	9	10	$k + 6$

Corollary 4.5. *For every $k \geq 6$ there is a d -dimensional k -crystal S such that $h(S) \geq 2^{d-2}(k + 6)$*

Proof. We can take the crystal from the example 4.1 and construct its direct product with \mathbb{Z}^{d-2} . Then the product of the empty polygon with $k + 6$ vertices with the cube $[0, 1]^{d-2}$ will give an empty polytope with $2^{d-2}(k + 6)$ vertices. \square

Thus, if $H(d, k)$ denotes the maximal Helly number of a d -dimensional k crystal (with large k), then $2^{d-2}(k + 6) \leq H(d, k) \leq k2^d$ and $H(d, k)$ is linear in k .

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APPENDIX A. PROOFS OF LEMMAS 4.3 AND 4.4

Proof of the lemma 4.3. Without loss of generality we can assume that three vertices of the quadrilateral are points $A = (0, 1)$, $B = (0, 0)$, and $C = (1, 0)$, and the fourth vertex $D = (x, y)$ satisfies $x, y > 0$ and $x + y > 1$.

If $x > 1$ and $y > 1$, then the parallelogram $ABCM$ with $M = (1, 1)$ will be inside $ABCD$. If $x > 1$ and $y \leq 1$, then the parallelogram $BCDM$ with $M = (x - 1, y)$ will be inside $ABCD$. If $x \leq 1$ and $y > 1$, then the parallelogram $ABMD$ with $M = (x, y - 1)$ will be inside $ABCD$. If $x \leq 1$ and $y \leq 1$ then $ABCD$ is either a parallelogram, or contains the parallelogram $AMCD$ with $M = (1 - x, 1 - y)$. \square

Proof of the lemma 4.4. If two of the segments XX' , YY' and ZZ' lie on line, then the lemma is trivial. Otherwise, without loss of generality we can assume that points have the following coordinates: $X = (0, 0)$, $X' = (1, 0)$, $Y = (0, 1)$, $Y' = (1, 1)$, $Z = (x, y)$, $Z' = (x + 1, y)$ for $y > 1$.

If $x > 0$, then Y' lies in $\text{conv}(X, X', Y, Z, Z')$. Otherwise, Y lies in $\text{conv}(X, X', Y', Z, Z')$. \square

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